ISRAEL JOURNAL OF MATHEMATICS 90 (1995), 301–323

GENERALIZATIONS IN THE LINEAR SEARCH PROBLEM

BY

VIC BASTON

Department of Mathematics
University of Southampton, Southampton SO9 5NH, U.K.
e-mail: vjdb@uk.ac.soton.maths

AND

ANATOLE BECK

Department of Mathematics
University of Wisconsin, Madison, Wisconsin 53706, U.S.A.
e-mail: beck@math.wisc.edu

ABSTRACT

A major part of the paper deals with the linear search problem in which the cost function is a strictly increasing convex function f satisfying $f(0) = 0$. It is shown that a number of results previously established for the case $f(t) = t^\alpha$ can be extended to the convex case; in particular a sufficient condition for the existence of a minimizing search strategy of a simple form is obtained for the convex case. Numerous results are obtained on the existence or otherwise of terminating and non-terminating optimal search strategies for cost functions already occurring in the literature.

1. Introduction and Notation

In a sequence of papers [1–7] spanning nearly thirty years, one of the authors (often with the help of co-authors) has investigated the following linear search problem.

A point on the real line is selected by means of a probability distribution F. The search for the point starts at zero and is made by a continuous motion with constant speed 1. The aim of the searcher is to minimize the expected cost where

Received April 13, 1993 and in revised form February 8, 1994

the cost is a function of the time taken to locate the point or equivalently the path length required to find the point.

In the earlier papers, the identity function was used for the cost but, in later papers, the analysis was generalized to cover the case when the cost is the αth power of the path length where $\alpha \geq 1$. Since the function $f(t) = t^\alpha$ for $t \geq 0$ and $\alpha \geq 1$ is a special case of a convex function, it is natural to consider whether analogous results can be obtained in a more general context. The purpose of this article is to show that a number of the results can be generalized to the case when the cost function is taken to be a strictly increasing convex function f satisfying $f(0) = 0$. **For the remainder of the paper f will be used solely to denote such a function.** Since f is convex, its derivatives from the right and from the left always exist and they will be denoted by f'_+ and f'_- respectively. In Section 3 we obtain a sufficient condition for the existence of a minimizing search strategy of a simple form while in Section 4 we show that, for the uniform distribution on a compact interval, the minimizing search strategy travels directly to one end-point and then directly to the other. The main result of Section 5 gives a necessary and sufficient condition for a distribution on a compact interval to have a minimizing search strategy which is non-terminating for the αth power case. Section 6 deals with a particular class of distributions called lopsided distributions and obtains various results on minimizing search strategies for these distributions.

To present the problem more mathematically we need to introduce some notation.

For a probability distribution F let $x^+ = x^+(F) = \sup\{t\colon F(t) < 1\}$ and $x^- = x^-(F) = \inf\{t\colon F(t) > 0\}$; we allow the possibilities $x^- = -\infty$ and $x^+ = \infty$. The linear search problem is only of interest when $x^- < 0 < x^+$ so we shall always assume that these inequalities hold. Further, as in previous papers on the topic and for the reasons given in them, we shall also assume that the probability functions $F(t)$ under consideration are continuous on the right for $t \geq 0$ and continuous on the left for $t \leq 0$. We denote the set of such functions by $\mathcal{F}$. A subset of $\mathcal{F}$ which plays an important role in the theory is one where the members are such that at least one of $\bar{F}^-(0)$ and $\bar{F}^+(0)$ is finite where

$$\bar{F}^-(0) = \limsup_{t \uparrow 0}(F(t) - F(0))/t \quad \text{and} \quad \bar{F}^+(0) = \limsup_{t \downarrow 0}(F(t) - F(0))/t.$$

We will denote the subset of $\mathcal{F}$ for which $\bar{F}^-(0)$ is finite by $\mathcal{F}^-$ and the subset for which $\bar{F}^+(0)$ is finite by $\mathcal{F}^+$.

A **generalized search strategy** is a doubly-infinite sequence $\mathbf{x} = \{x_i\}_{i=-\infty}^{\infty}$ satisfying

$$\cdots \leq x_{2i+2} \leq x_{2i} \leq \cdots \leq 0 \leq \cdots \leq x_{2i-1} \leq x_{2i+1} \leq \cdots.$$

The set of all such strategies is denoted by $\mathcal{X}$. It is clear that if there is an n such that x_n and x_{n+1} are x^- and x^+ in either order then the value of the expected cost of locating the point is the same regardless of the values of succeeding entries. We shall allow x_n to be x^- or x^+ even when they are infinite. When there is an integer i such that $x_i = x^-$ or x^+ a strategy is said to be **terminating**; otherwise it is said to be **non-terminating**. Intuitively by choosing $\mathbf{x} \in \mathcal{X}$ the searcher employs a path in which, for each integer r, he goes from x_r to x_{r+1}.

If $\mathbf{x} \in \mathcal{X}$ and there is an integer m such that $x_i = 0$ for all $i < m$, we say that $\mathbf{x}$ is a **standard search strategy**. Clearly, for any standard search strategy $\mathbf{x}$, the subscripts can be renumbered so that the first non-zero term is x_1. We put

$$\mathcal{Y}^+ = \{\mathbf{x} \in \mathcal{X}\colon x_i = 0 \text{ for } i \leq 0,\ x_{2i-1} > 0 \text{ and } x_{2i} < 0 \text{ for } i > 0\},$$
$$\mathcal{Y}^- = \{\mathbf{x} \in \mathcal{X}\colon x_i = 0 \text{ for } i \leq 0,\ x_{2i-1} < 0 \text{ and } x_{2i} > 0 \text{ for } i > 0\}$$

and

$$\mathcal{Y} = \mathcal{Y}^+ \cup \mathcal{Y}^-.$$

For $\mathbf{x} \in \mathcal{X}$ and a real number t, $X(\mathbf{x}, t)$ is defined as follows: for t lying between x_{n-1} and x_{n+1}, $X(\mathbf{x}, t) = |t| + 2s_n(\mathbf{x})$ where $s_n(\mathbf{x}) = \sum_{i=-\infty}^{n} |x_i|$. Thus $X(\mathbf{x}, t)$ is the path length taken to reach the point t when using $\mathbf{x}$.

For $\mathbf{x} \in \mathcal{X}$ the **expected cost function** $X_f(\mathbf{x})$ is then defined by

$$\int_{-\infty}^{\infty} f(X(\mathbf{x}, t))dF(t).$$

A trivial argument shows that $\inf_{\mathbf{x}\in\mathcal{X}} X_f(\mathbf{x}) = \inf_{\mathbf{x}\in\mathcal{Y}} X_f(x)$ and in future we shall use μ to denote either of these expressions. If there is no search strategy for which X_f is finite we adopt the convention that μ is ∞ and every search strategy is minimizing.

2. Preliminary results

In the particular case when f is the αth power of the path length with $\alpha \geq 1$, μ is finite whenever $\int_{-\infty}^{\infty} f(|t|)dF$ is finite. That this does not carry over directly to our more general situation is easily seen by taking $f(t) = \exp(t^2)$ and F the symmetric distribution having density function ρ of the form $\rho(t) = k\exp(-t^2 - t)$ for $t \geq 0$ and appropriate k. However the following lemma shows that there is a result of the same form for the general case.

LEMMA 2.1: *If $\int_{-\infty}^{\infty} f(9|t|)dF(t) < \infty$, then μ is finite. Further we have*

$$\mu \leq \int_{-\infty}^{\infty} f(9|t|)dF(t).$$

Proof: Straightforward generalization of the proof for the case $f(t) = t$ in [1]. ∎

Remark 2.2: If x^+ and x^- are both finite, then μ is finite.

Proof: Trivial.

LEMMA 2.3: *Let $F \in \mathcal{F}$, μ be finite and $(\mathbf{x^{(n)}})$ a sequence in $\mathcal{Y}$ such that $X_f(\mathbf{x^{(n)}}) \to \mu$. Then there is a subsequence $(\mathbf{y^{(n)}})$ of $(\mathbf{x^{(n)}})$ such that one of the following holds:*

- (α) *There exists a sequence (y_i) of real numbers such that $y_i^{(n)} \to y_i$ as $n \to \infty$ for all positive integers i.*
- (β) *There is a positive integer k and real numbers y_i $(i = 1, \dots, k-1)$ such that $x^- < y_i < x^+$ for $i < k-1$, $y_{k-1} \in \{x^-, x^+\}$ and $y_i^{(n)} \to y_i$ as $n \to \infty$ for $i \leq k-1$.*

Proof: Let $(\mathbf{x^{(n)}})$ be a sequence in $\mathcal{Y}$ such that $X_f(\mathbf{x^{(n)}}) \to \mu$ as $n \to \infty$. By taking a subsequence we may assume $X_f(\mathbf{x^{(n)}}) < \mu + 1$ for all n. Two cases arise.
(α) For all positive integers i there is a b_i such that $x_i^{(n)} \in [-b_i, b_i]$ for all n. In this case the result follows via a standard diagonalization procedure.
(β) There is a positive integer i such that, for all b, there is an n such that $|x_i^{(n)}| > b$. Let k be the smallest integer i for which this holds. Then, by taking a subsequence, we may assume that $(\mathbf{x^{(n)}})$ also satisfies $x_i^{(n)} \to x_i$ for $0 \leq i \leq k-1$. We show that $x_{k-1} = x^-$ or x^+.

Firstly suppose $x_{k-1}^{(n)} \leq 0$ and $x_{k-1} \neq x^-$. Then, by taking a subsequence, we may assume that $F(x_{k-1}^{(n)}) > F(x_{k-1})/2 > 0$ for all n. Put $A = 2(\mu+1)/F(x_{k-1})$

and take B such that $f(t) > A$ for $t > B$, then there is a positive integer N such that $x_k^{(N)} > B$. We then have

$$X_f(\mathbf{x^{(N)}}) \geq \int_{-\infty}^{x_{k-1}^{(N)}} f(X,t)dF \geq \int_{-\infty}^{x_{k-1}^{(N)}} f(2x_k^{(N)})dF \geq f(2B)F(x_{k-1}^{(N)})$$
$$> AF(x_{k-1}^{(N)}) = 2(\mu+1)$$

and we have a contradiction. Hence, if $x_{k-1}^{(n)} \leq 0$, we have $x_{k-1} = x^-$.

Similar arguments show that $x_{k-1} = x^+$ when $x_{k-1}^{(n)} \geq 0$. ∎

When (α) or (β) of Lemma 2.3 are satisfied by a sequence $(\mathbf{y^{(n)}})$ in $\mathcal{Y}$ we write $\mathbf{y^{(n)}} \to \mathbf{y}$.

3. Sufficient conditions for a minimizing standard search strategy

In this section it is shown that there is always a minimizing standard search strategy when $F \in \mathcal{F}^- \cup \mathcal{F}^+$. The corresponding result for the case when $f(t) = t^\alpha$ where $\alpha \geq 1$ and x^-, x^+ are both infinite is given by Theorem 9 in [4]. The proof of that theorem used a previous lemma (Lemma 7) which in turn appealed to a previous lemma (Lemma 4). Unfortunately Lemma 4 cannot be generalized to the present context as the following example shows.

Example: Let $f(t) = \exp(t^2)$ and F be the discrete probability distribution which, for an appropriate k, has probabilities $kj^{-2}\exp\{-(3.2^{j-1}-2)^2\}$ at the points $(-2)^{j-1}$ for $j = 1, 2, \ldots$. Then, for $\mathbf{y} \in \mathcal{Y}$ defined by $y_i = (-2)^{i+1}$ for $i = 1, 2, \ldots,$ we have

$$X_f(\mathbf{y}) = k\sum_{j=1}^{\infty} j^{-2} < \infty$$

but, for $0 < \eta \leq 1$,

$$\int_{-\infty}^{\infty} f(X_f(\mathbf{y},t)+\eta)dF(t) = k\sum_{j=1}^{\infty} j^{-2}\exp\{\eta(3.2^{j-1}-2)+\eta^2\}$$

which diverges.

LEMMA 3.1: *Let $F \in \mathcal{F}^-$. Then there is a $K > 0$ satisfying $2F(K) < 1 + F(0)$ such that, whenever $\mathbf{x} \in \mathcal{Y}^+$ and $x_2, x_3 \in (-K, K)$, we have $X_f(\mathbf{y}) \leq X_f(\mathbf{x})$ where $\mathbf{y} \in \mathcal{Y}^+$ is given by $y_i = x_{i+2}$.*

Proof: Since $F \in \mathcal{F}^-$ there is a positive number D such that $\bar{F}^-(0) < D$. Choose $K > 0$ such that

$$\begin{aligned} K &< (1/D)F(-K), \\ 2F(K) &< 1 + F(0), \\ K &< \min\{x^+, |x^-|\} \end{aligned}$$

and

$$F(0) - F(t) < (-t)D \quad \text{for all } -K \le t < 0.$$

(Note that the first and second inequalities can be satisfied for arbitrarily small values of K since $0 < F(0) < 1$ and F is continuous at 0 while the fourth can be satisfied because $\bar{F}^-(0) < D$.)

Let $\mathbf{x} \in \mathcal{Y}^+$ with $x_1 < x^+$ and $x_2 > x^-$. Define $\mathbf{y} \in \mathcal{Y}^+$ by $y_i = x_{i+2}$ then

$$\begin{aligned} D_f(\mathbf{x},\mathbf{y}) &= X_f(\mathbf{x}) - X_f(\mathbf{y}) \\ &= \sum_{i=0}^{\infty} \int_{x_{2i+1}}^{x_{2i+3}} f(2s_{2i+2}(\mathbf{x}) + t) - f(2s_{2i+2}(\mathbf{x}) + t - 2x_1 + 2x_2)dF(t) \\ &\quad + \int_{x_2}^{0} f(|t| + 2x_1) - f(|t| + 2x_3)dF(t) \\ &\quad + \sum_{i=1}^{\infty} \int_{x_{2i+2}}^{x_{2i}} f(|t| + 2s_{2i+1}(\mathbf{x})) - f(|t| + 2s_{2i+1}(\mathbf{x}) - 2x_1 + 2x_2)dF \\ &\ge f(2x_1 - 2x_2)(1 - F(x_1)) - 2(x_3 - x_1)f'_-(2x_3 - x_2)(F(0) - F(x_2)) \\ &\quad + 2(x_1 - x_2)f'_+(2x_3 - x_2)F(x_2) \\ &\ge 2f'_-(2x_3 - x_2)\{(-x_2)F(x_2) - (x_3 - x_1)(F(0) - F(x_2))\}. \end{aligned}$$

Now, for $-K < x_2$ and $x_3 < K$,

$$\begin{aligned} (-x_2)F(x_2) \ge (-x_2)F(-K) > -x_2DK > \{F(0) - F(x_2)\}K \\ \ge (x_3 - x_1)\{F(0) - F(x_2)\} \end{aligned}$$

and the result follows. ∎

LEMMA 3.2: *Let $F \in \mathcal{F}^-$ and μ be finite then, for the $K > 0$ of the previous lemma, we have that, for all $\mathbf{x} \in \mathcal{Y}^+$ satisfying $X_f(\mathbf{x}) < \mu + 1$, there is a $\mathbf{y} \in \mathcal{Y}^+$ such that $X_f(\mathbf{y}) \le X_f(\mathbf{x})$ and at least one of $y_2 \le -K$ and $y_3 \ge K$ holds.*

Proof: Since $f(t) \to \infty$ as $t \to \infty$, we can take A such that

$$f(t) > \frac{2(\mu + 1)}{1 - F(0)} \quad \text{for } t \ge A.$$

Let $\mathbf{x} \in \mathcal{Y}^+$ satisfy $X_f(\mathbf{x}) < \mu + 1$. If $x_2 \leq -K$ or $x_3 \geq K$ there is nothing to do. Thus suppose $-K < x_2$ and $x_3 < K$; then, by the previous lemma, there is a $\mathbf{w}^{(\mathbf{1})} \in \mathcal{Y}^+$ such that $w_i^{(1)} = x_{i+2}$ and $X_f(\mathbf{w}^{(\mathbf{1})}) \leq X_f(\mathbf{x})$. If $w_2^{(1)} \leq -K$ or $w_3^{(1)} \geq K$ we are through. Otherwise repeat the process until we reach a $\mathbf{w}^{(\mathbf{r})} \in \mathcal{Y}^+$ such that $w_i^{(r)} = x_{i+2r}$, $X_f(\mathbf{w}^{(\mathbf{r})}) \leq X_f(\mathbf{x})$ and $w_2^{(r)} \leq -K$ or $w_3^{(r)} \geq K$. Such an r must be reached since, for any positive integer $k \geq A/x_1$, we have

$$\begin{aligned} X_f(\mathbf{x}) &\geq \sum_{j=1}^{\infty} \int_{x_{2k+2j+1}}^{x_{2k+2j+3}} f(kx_1)dF \geq (1 - F(x_{2k+3}))2(\mu+1)/(1-F(0)) \\ &= (1 - F(w_3^{(k)}))2(\mu+1)/(1-F(0)) > \mu + 1 \text{ when } w_3^{(k)} < K. \qquad \blacksquare \end{aligned}$$

LEMMA 3.3: *Let $F \in \mathcal{F}^-$, μ finite and $(\mathbf{x}^{(\mathbf{n})})$ be a sequence in $\mathcal{Y}^+$ such that $X_f(\mathbf{x}^{(\mathbf{n})}) \to \mu$, then there is a sequence $(\mathbf{y}^{(\mathbf{n})})$ in $\mathcal{Y}$ such that $X_f(\mathbf{y}^{(\mathbf{n})}) \to \mu$ and $\mathbf{y}^{(\mathbf{n})} \to \mathbf{y}$ where, for all positive integers i, $y_i \neq 0$.*

Proof: By Lemmas 3.2 and 2.3 we may take the $(\mathbf{x}^{(\mathbf{n})})$ in the statement to be a sequence in $\mathcal{Y}^+$ such that $\mathbf{x}^{(\mathbf{n})} \to \mathbf{x}$ and $x_2^{(n)} \leq -K$ or $x_3^{(n)} \geq K$ for all n. Thus at least one of $x_2 < 0$ and $x_3 > 0$ holds.

If $x_2 = 0$ then $0 < x_3 < \infty$. Define $\mathbf{y}^{(\mathbf{n})} \in \mathcal{Y}^+$ by $y_i^{(n)} = x_{i+2}^{(n)}$ for $i \geq 1$, then

$$\begin{aligned} X_f(\mathbf{x}^{(\mathbf{n})}) - X_f(\mathbf{y}^{(\mathbf{n})}) &\geq \int_{x_2^{(n)}}^{0} f(2x_1^{(n)} + |t|) - f(2x_3^{(n)} + |t|)dF \\ &\geq -f(2x_3 + 1)(F(0) - F(x_2^{(n)})) \end{aligned}$$

for large enough n. Thus $X_f(\mathbf{y}^{(\mathbf{n})}) \leq X_f(\mathbf{x}^{(\mathbf{n})}) + f(2x_3+1)(F(0) - F(x_2^{(n)})) \to \mu$ as $n \to \infty$ since F is continuous at 0 and $x_2^{(n)} \to x_2 = 0$. Hence $X_f(\mathbf{y}^{(\mathbf{n})}) \to \mu$ as $n \to \infty$. Since $y_1^{(n)} = x_3^{(n)} \to x_3 > 0$ we are through if $y_2 = x_4 \neq 0$.

Thus let $y_2 = 0$. Then $y_1 = x_3$ and $F(y_1) \neq 1$; further y_3 is finite. Define $\mathbf{z}^{(\mathbf{n})} \in \mathcal{Y}^+$ by $z_i^{(n)} = y_{i+2}^{(n)}$ for $i \geq 1$; then

$$\begin{aligned} X_f(\mathbf{y}^{(\mathbf{n})}) - X_f(\mathbf{z}^{(\mathbf{n})}) &\geq \int_{y_2^{(n)}}^{0} f(2y_1^{(n)} + |t|) - f(2y_3^{(n)} + |t|)dF \\ &\quad + \int_{y_1^{(n)}}^{\infty} f(2y_1^{(n)} + t) - f(t)dF \\ &\geq -f(2y_3^{(n)} + |y_2^{(n)}|)(F(0) - F(y_2^{(n)})) \\ &\quad + 2y_1^{(n)} f'_+(y_1^{(n)})(1 - F(y_1^{(n)})). \end{aligned}$$

Since $y_1^{(n)} \to y_1 > 0$ and $F(y_1) \neq 1$, the second term is bounded away from zero as $n \to \infty$ whereas the first term tends to zero as $n \to \infty$ because F is continuous at 0 and $y_2^{(n)} \to 0$ and $2y_3^{(n)} + |y_2^{(n)}| \leq 2y_3 + |y_2| + 1$ for sufficiently large n. Hence, for sufficiently large n, $X_f(\mathbf{z^{(n)}})$ would be strictly less than μ which is an impossibility.

Thus we may take $x_2 \neq 0$. Suppose $x_1 = 0 = x_3 = \cdots = x_{2k-1}$ but $x_{2k+1} \neq 0$ for some positive integer k then x_{2k} is finite. Consider $\mathbf{y^{(n)}} \in \mathcal{Y}^-$ given by $y_i^{(n)} = x_{i+2k-1}^{(n)}$ for $i \geq 1$, then

$$X_f(\mathbf{x^{(n)}}) - X_f(\mathbf{y^{(n)}}) \geq -\int_0^{x_{2k-1}^{(n)}} f(2|x_{2k}^{(n)}| + t)dF$$
$$\geq -f(2|x_{2k}^{(n)}| + x_{2k-1}^{(n)})(F(x_{2k-1}^{(n)}) - F(0)).$$

Thus, for sufficiently large n,

$$X_f(\mathbf{y^{(n)}}) \leq X_f(\mathbf{x^{(n)}}) + f(2|x_{2k}| + 1)(F(x_{2k-1}^{(n)}) - F(0)) \to \mu \text{ as } n \to \infty.$$

Hence $X_f(\mathbf{y^{(n)}}) \to \mu$ and $y_i^{(n)} \to x_{i+2k-1} \neq 0$ for any positive integer i. ∎

LEMMA 3.4: *Let $F \in \mathcal{F}^-$ and μ be finite, then there is a sequence $(\mathbf{y^{(n)}})$ in $\mathcal{Y}$ such that $X_f(\mathbf{y^{(n)}}) \to \mu$ and $\mathbf{y^{(n)}} \to \mathbf{y}$ where, for all positive integers i, $y_i \neq 0$. Furthermore $|y_{i+2}| > |y_i|$ for all positive integers i for which $x^- < y_i < x^+$.*

Proof: Let $(\mathbf{x^{(n)}})$ be a sequence in $\mathcal{Y}$ satisfying $X_f(\mathbf{x^{(n)}}) \to \mu$ and $\mathbf{x^{(n)}} \to \mathbf{x}$ as $n \to \infty$. If $x_1^{(n)} > 0$ for an infinite number of n, the result follows from the previous lemma. Hence we may assume that $x_1^{(n)} < 0$ for all n.

Suppose $x_1 = 0$ then x_2 is finite. Let $\mathbf{y^{(n)}} \in \mathcal{Y}^+$ be given by $y_i^{(n)} = x_{i+1}^{(n)}$ for $i \geq 1$; then

$$X_f(\mathbf{y^{(n)}}) - X_f(\mathbf{x^{(n)}}) \leq \int_{x_1^{(n)}}^0 \{f(2x_2^{(n)} + |t|) - f(|t|)\}dF$$
$$\leq 2x_2^{(n)} f'_-(2x_2^{(n)} + |x_1^{(n)}|)(F(0) - F(x_1^{(n)}))$$
$$\leq (2x_2 + 1)f'_-(2x_2 + 1)(F(0) - F(x_1^{(n)})$$

for large enough n.

Since $x_1^{(n)} \to 0$ and F is continuous at 0, we have $X_f(\mathbf{y^{(n)}}) \to \mu$ and the result follows by the previous lemma.

Hence let $x_1 < 0$. Suppose $x_2 = 0$ then $x_1 \neq x^-$ and x_3 is finite. Let $\mathbf{y^{(n)}} \in \mathcal{Y}^-$ be given by $y_i^{(n)} = x_{i+2}^{(n)}$ for $i \geq 1$, then

$$\begin{aligned}
X_f(\mathbf{y^{(n)}}) - X_f(\mathbf{x^{(n)}}) &\leq \int_0^{x_2^{(n)}} \{f(2|x_3^{(n)}| + |t|) - f(2|x_1^{(n)}| + |t|)\}dF \\
&\quad + \sum_{i=0}^{\infty} \int_{x_{2i+3}^{(n)}}^{x_{2i+1}^{(n)}} f(|t|) - f(|t| + 2|x_1^{(n)}|)dF \\
&\leq 2(|x_3^{(n)}| - |x_1^{(n)}|)f'_-(2|x_3^{(n)}| + x_2^{(n)})(F(x_2^{(n)}) - F(0)) \\
&\quad - 2|x_1^{(n)}|f'_+(|x_1^{(n)}|)(F(x_1^{(n)}).
\end{aligned}$$

Since F is continuous at 0, the first term can be made arbitrarily small by taking n large enough. However, the second term is strictly negative and bounded away from zero as $n \to \infty$ because $x^- < x_1 < 0$ so we have a contradiction to the supposition that $X_f(\mathbf{x^{(n)}}) \to \mu$.
Hence $x_2 \neq 0$ and the first part of the lemma follows.

Suppose there is a positive integer k such that $x^- < y_k < x^+$ and $y_{k+2} = y_k$. Note that this implies that y_{k+3} is finite. We shall only treat the case $y_k > 0$ as the case $y_k < 0$ follows by analogous arguments. Put $\eta = 1 - F(y_k) > 0$ and $\epsilon = \eta f(2y_k)/8$. Since f is uniformly continuous in $[0, 2s_{k+3}(\mathbf{y}) + 1]$, there is a $\delta > 0$ such that $|f(u) - f(v)| < \epsilon$ whenever $|u - v| < \delta$ and u, v lie in $[0, 2s_{k+3}(\mathbf{y}) + 1]$. We may clearly suppose $\delta < 1$. Now choose N so that $\mathbf{y^{(N)}}$ satisfies

$$\begin{aligned}
X_f(\mathbf{y^{(N)}}) &< \mu + \epsilon/2, & (1 + F(y_k))/2 &> F(y_{k+2}^{(N)}), \\
|y_{k+2}^{(N)} - y_k^{(N)}| &< \delta/4, & |y_{k+1}^{(N)} - y_{k+1}| &< \delta/4, \\
|y_k^{(N)} - y_k| &< \delta/4, & |s_k(y^{(N)}) - s_k(y)| &< \delta/4.
\end{aligned}$$

Take $\mathbf{z}$ in $\mathcal{Y}$ defined by

$$z_i = \begin{cases} y_i^{(N)} & \text{if } i < k; \\ y_{i+2}^{(N)} & \text{if } i \geq k. \end{cases}$$

Hence

$$\begin{aligned}
X_f(\mathbf{z}) - X_f(\mathbf{y^{(N)}}) &\leq \int_{y_{k+1}^{(N)}}^{y_{k-1}^{(N)}} f(2s_k(\mathbf{z}) + |t|) - f(2s_k(\mathbf{y^{(N)}}) + |t|)dF \\
&\quad + \int_{y_{k+2}^{(N)}}^{x^+} f(2s_{k+1}(\mathbf{z}) + t) - f(2s_{k+3}(\mathbf{y^{(N)}}) + t)dF \\
&\leq \epsilon(F(y_{k-1}^{(N)}) - F(y_{k+1}^{(N)})) - f(2y_k^{(N)})(1 - F(y_{k+2}^{(N)})) \\
&\leq \epsilon - (f(2y_k) - \epsilon)(1 - F(y_{k+2}^{(N)})) \leq -\epsilon.
\end{aligned}$$

Thus $X_f(\mathbf{z}) \leq \mu - \epsilon/2$ so we have a contradiction and the lemma follows. ∎

THEOREM 3.5: *Let $F \in \mathcal{F}^- \cup \mathcal{F}^+$. Then there is a minimizing standard search strategy.*

Proof: We may assume without loss of generality that $F \in \mathcal{F}^-$; the other case is similar *mutatis mutandis*. Suppose the result is false. Then $\mu < \infty$. By the previous lemma we may take a sequence $(\mathbf{x^{(n)}})$ in $\mathcal{Y}$ such that $X_f(\mathbf{x^{(n)}}) \to \mu$ and $\mathbf{x^{(n)}} \to \mathbf{x}$ where $|x_i| \neq 0$ for every positive integer i and $|x_{i+2}| > |x_i|$ for all non-negative integers i for which $x^- < x_i < x^+$. Thus we may in a natural way consider $\mathbf{x}$ as a member of $\mathcal{Y}$. Let $\alpha = \min\{\mu+1, X_f(\mathbf{x})\}$. Then $\delta = (\alpha-\mu)/4 > 0$ and there is a least positive integer M such that

$$\sum_{n=-1}^{M-2} \left| \int_{x_n}^{x_{n+2}} f(X(\mathbf{x},t))dF \right| > \mu + 3\delta.$$

Clearly x_i is finite for $i \leq M-1$. There is the possibility that x_M is infinite; when this happens, in the analysis that follows x_M is to be interpreted as a real number x'_M satisfying

$$|x'_M| > |x_{M-2}| + 1, \quad |F(x'_M) - F(x_M)| < \delta/4MC$$

and

$$\sum_{n=-1}^{M-3} \left| \int_{x_n}^{x_{n+2}} f(X(\mathbf{x},t))dF \right| + \left| \int_{x_{M-2}}^{x'_M} f(X(\mathbf{x},t))dF \right| > \mu + 3\delta$$

while $s_M(\mathbf{x})$ is to be interpreted as $s_{M-1}(\mathbf{x}) + x'_M$. Hence in all cases $s_M(\mathbf{x})$ is finite.

Since f is uniformly continuous in $[0,\ 2s_M(\mathbf{x})+1]$, there is an $\epsilon > 0$ such that

$$|f(u) - f(v)| < \delta/(4M) \text{ when } |v-u| < \epsilon \text{ and } u \text{ and } v \text{ both lie in } [0,\ 2s_M(x)+1].$$

We may clearly take $\epsilon < \min\left\{1, \min_{-1\leq i\leq M-2}\{|x_{i+2} - x_i|\}\right\}/2$.

Let $C = f(2s_M(\mathbf{x})+1)$ and take $\mathbf{x^{(N)}}$ so that $X_f(\mathbf{x^{(N)}}) < \mu + \delta$, and, for all $i \leq M-1$,

$$|x_i^{(N)} - x_i| < \epsilon/\{4M\},$$

and

$$|F(x_i) - F(x_i^{(N)})| < \delta/\{4MC\} \quad \text{when } |x_i^{(N)}| \geq |x_i|.$$

Note that the first condition implies that

$$\max\{|x_i|,\ |x_i^{(N)}|\} < \min\{|x_{i+2}|,\ |x_{i+2}^{(N)}|\} \quad \text{for } i = 0, \ldots, M-2$$

and also that $|s_r(\mathbf{x}^{(\mathbf{N})}) - s_r(\mathbf{x})| < \epsilon/2$ for $r = 1, 2, \ldots, M-1$.

Put $I(i,N) = \int_{x_i}^{x_{i+2}} f(X(\mathbf{x},t)) - f(X(\mathbf{x}^{(\mathbf{N})},t))dF$.

If $-1 \le i \le M-2$ and $0 < x_i \le x_i^{(N)}$, we have

$$\begin{aligned}
I(i,N) \le & \int_{x_i}^{x_i^{(N)}} f(2s_{i+1}(\mathbf{x})+t) - f(2s_{i-1}(\mathbf{x}^{(\mathbf{N})})+t)dF \\
& + \int_{x_i{}^{(N)}}^{x_{i+2}} f(2s_{i+1}(\mathbf{x})+t) - f(2s_{i+1}(\mathbf{x}^{(\mathbf{N})})+t)dF \\
& \le C(F(x_i^{(N)}) - F(x_i)) + (F(x_{i+2}) - F(x_i^{(N)}))\delta/4M \\
& \le \delta/2M
\end{aligned}$$

and if $-1 \le i \le M-2$ and $0 \le x_i^{(N)} < x_i$, we have

$$\begin{aligned}
I(i,N) \le & \int_{x_i}^{x_{i+2}} f(2s_{i+1}(\mathbf{x})+t) - f(2s_{i+1}(\mathbf{x}^{(\mathbf{N})})+t)dF \\
& \le (F(x_{i+2}) - F(x_i))\delta/4M \le \delta/4M.
\end{aligned}$$

For $-1 \le i \le M-2$ and $x_i < 0$ we have, by analogous arguments,

$$\int_{x_{i+2}}^{x_i} f(X(\mathbf{x},t)) - f(X(\mathbf{x}^{(\mathbf{N})},t))dF \le \delta/2M.$$

It follows that

$$\begin{aligned}
X_f(\mathbf{x}^{(\mathbf{N})}) \ge & \sum_{i=-1}^{M-2} \left| \int_{x_i}^{x_{i+2}} f(X(\mathbf{x}^{(\mathbf{N})},t))dF \right| \\
\ge & \sum_{i=-1}^{M-2} \left| \int_{x_i}^{x_{i+2}} f(X(\mathbf{x},t))dF \right| - \delta/2 > \mu + 2\delta.
\end{aligned}$$

This contradiction proves the theorem. ∎

4. Uniform distribution

In this section we take positive numbers a and b and consider the uniform distribution F defined by $F'(t) = 1/(b+a)$ for all $-a < t < b$ and $F'(t) = 0$ otherwise. It will also be assumed in this section that f is differentiable. We will show that for this case a minimizing search strategy first visits one of the end-points and then the other. Our results therefore generalize those in Section 2 of [7]. However, although the statements of our lemmas are very similar to those in [7], our methods of proof differ markedly from the ones in that paper. In particular our proofs will use some standard properties of convex functions (see [9]).

For positive numbers x, a and u, let

$$g(x) = \int_0^x f(t)dt \quad \text{and} \quad h_a(u) = g(a+u) - g(u).$$

Now g is convex because f is increasing. Further $g''(x) = f'(x)$. Hence $h_a'(u) = f(a+u) - f(u) > 0$ because f is strictly increasing and $h_a''(u) = f'(a+u) - f'(u) > 0$ since f' is increasing because f is convex. Thus h_a is a convex strictly increasing function. We will be applying the following result (see [8] p.164) to the functions h_a and h_b for appropriate a and b.

LEMMA 4.1: *If h is a convex increasing function, $u_1 \geq u_2 \geq \cdots \geq u_n$, $v_1 \geq v_2 \geq \cdots \geq v_n$ and*

$$\sum_{r=1}^{k} v_r \leq \sum_{r=1}^{k} u_r \qquad \text{for } k = 1, 2, \ldots, n,$$

then

$$\sum_{r=1}^{n} h(v_r) \leq \sum_{r=1}^{n} h(u_r).$$

We also require the following lemmas.

LEMMA 4.2: *If $0 < a < b$, then*

$$\int_0^a f(t)dt + \int_0^b f(2a+t)dt \leq \int_0^b f(t)dt + \int_0^a f(2b+t)dt.$$

Proof: Since $a < b$, by Lemma 4.1 we have

$$h_a(a+b) + h_a(b) \leq h_a(2b) + h_a(a).$$

Putting in terms of g and rearranging we obtain

$$g(a) + \{g(2a+b) - g(2a)\} \leq g(b) + \{g(2b+a) - g(2b)\}$$

and this is effectively the inequality in the statement of the lemma. ∎

LEMMA 4.3: *Whenever $0 < a, b < c$ the following inequality holds:*

$$\int_0^b f(t)dt + \int_0^c f(2b+t)dt < \int_0^a f(t)dt + \int_0^b f(2a+t)dt + \int_a^c f(2a+2b+t)dy.$$

Proof: (i) Suppose $0 < a < b < c$. Then, by Lemma 4.1, we have

$$h_a(a+2b+c) + h_a(2b+c) + h_a(a+b) \geq h_a(2a+2b) + h_a(a+2b) + h_a(2b)$$

so

$$h_a(a+2b+c) + h_a(2b+c) + h_a(a+b) + h_a(b) > h_a(2a+2b) + h_a(a+2b) + h_a(2b) + h_a(a).$$

Putting in terms of g and rearranging we obtain

$$g(b) + g(2b+c) - g(2b) < g(a) + g(2a+b) - g(2a) + g(2a+2b+c) - g(3a+2b)$$

and this is effectively the inequality in the statement of the lemma.
(ii) Now suppose $0 < b \leq a < c$. As in [7] the inequality in the statement of the lemma follows from Lemma 4.2 with the roles of a and b reversed and the fact that

$$\int_a^c f(2b+t)dt < \int_a^c f(2a+2b+t)dt$$

which holds because f is strictly increasing.

The lemma now follows immediately from (i) and (ii). ∎

LEMMA 4.4: *If $0 < a < c \leq b \leq d$, then*

$$\int_0^c f(t)dt + \int_0^d f(2c+t)dt < \int_0^a f(t)dt + \int_0^b f(2a+t)dt + \int_a^c f(2a+2b+t)dt + \int_b^d f(2a+2b+2c+t)dt.$$

Proof: Since $0 < a < c \leq b \leq d$, it is easy to verify from Lemma 4.1 that

$$h_b(2a+b+2c+d) + h_b(a+2c+d) + h_b(a+b+c) + h_b(a+c) \geq h_b(2a+2b+2c) + h_b(a+b+2c) + h_b(2a+b) + h_b(2c)$$

and also that

$$h_a(a+2b+2c) + h_a(b+2c) + h_a(2a+2b) \leq h_a(a+b+2c+d) + h_a(2c+d) + h_a(a+2b+c).$$

(Note that, in the latter case, we do not know whether or not $b+2c$ is greater than or equal to $2a+b$ and so we have to check the cases separately.)

The last inequality gives

$$h_a(a+2b+2c)+h_a(b+2c)+h_a(2a+2b)+h_a(a)$$
$$< h_a(a+b+2c+d)+h_a(2c+d)+h_a(a+2b+c)+h_a(c).$$

From this inequality and the one involving h_b we have

$$\{h_b(2a+b+2c+d)+h_a(a+b+2c+d)+h_b(a+2c+d)+h_a(2c+d)\}$$
$$+\{h_a(a+2b+c)+h_b(a+b+c)+h_b(a+c)+h_a(c)\}$$
$$<\{h_b(2a+2b+2c)+h_a(a+2b+2c)+h_b((a+b+2c)$$
$$+h_a(b+2c)+h_b(2c)\}+\{h_a(2a+2b)+h_b(2a+b)\}+h_a(a).$$

Putting in terms of g we obtain

$$g(2a+2b+2c+d)-g(2c+d)+g(2a+2b+c)-g(c)$$
$$> g(2a+3b+2c)-g(2c)+g(3a+2b)-g(2a+b)+g(2a)-g(a).$$

Thus

$$\{g(2a+2b+2c+d)-g(2a+3b+2c)\}+\{g(2a+2b+c)-g(3a+2b)\}$$
$$+\{g(2a+b)-g(2a)\}+g(a)$$
$$>\{g(2c+d)-g(2c)\}+g(c)$$

which is effectively the inequality in the statement of the lemma. ∎

THEOREM 4.5: *Let F be the uniform distribution over the finite interval $[-a,b]$ where a and b are positive.*

(i) *If $a>b$, then $X_f(\mathbf{x})$ is minimized by $\mathbf{w}$ where $\mathbf{w}$ first goes to b and then to $-a$.*

(ii) *If $a\le b$, then $X_f(\mathbf{x})$ is minimized by $\mathbf{w}$ where $\mathbf{w}$ first goes to $-a$ and then to b.*

Proof: By Theorem 3.5 there is a minimizing strategy $\mathbf{x}$ in $\mathcal{Y}$. We cannot have $x_1\in(-a,b)$ and $|x_2|<|x_3|$, since in that case by either Lemma 4.3(i) (if $|x_1|<|x_2|$) or Lemma 4.3(ii) (if $|x_1|\ge|x_2|$), the substitution of 0 for x_1 would reduce $X_f(\mathbf{x})$. If $x_1\in(-a,b)$ and $|x_2|\ge|x_3|$, then by Lemma 4.4, the substitution of 0 for x_1 and x_2 would reduce $X_f(\mathbf{x})$. Thus, in all cases, $X_f(\mathbf{x})$ is not minimized by $x_1\in(-a,b)$. It follows that x_1 is one of the end-points and so x_2 is necessarily the other end-point. By Lemma 4.2, x_1 must be the end-point with the smaller absolute value and the theorem follows. ∎

5. Skinny distributions

In Section 4 we saw that, when F is the uniform distribution on a finite interval, there is a minimizing search strategy which terminates. On the other hand it was shown in [7] that, if F is the triangular distribution on $[-1,1]$ and $f(t)=t^\alpha$ where $\alpha \geq 1$, then every minimizing search strategy is non-terminating. In this section we obtain a necessary and sufficient condition for a compact distribution to have a minimizing search strategy that terminates for the case when $f(t)=t^\alpha$ for $\alpha \geq 1$. In doing so we also obtain a weaker result for the general case. We require the following definition.

Definition: Let F be a distribution with at least one of x^- and x^+ real. If x^- is real we say that F is **skinny at** x^- if

$$\liminf_{t \downarrow x^-} F(t)/(t-x^-)=0.$$

If x^+ is real we say that F is **skinny at** x^+ if

$$\liminf_{t \uparrow x^+}(1-F(t))/(x^+-t)=0.$$

If x^- is real we also say that F is **fat** at x^- if $F(t)/(t-x^-)$ is bounded away from 0 for $x^- < t < x^- + 1$. Similarly for F is fat at x^+. Clearly F is fat at x^- (or x^+) if and only if it is not skinny there.

THEOREM 5.1: *Let F be compact with support $[a,b]$, where $a<0<b$ and skinny at both a and b. Then every minimizing search strategy is non-terminating.*

Proof: Let F be skinny at a, and suppose that there is an optimal search strategy $\mathbf{x}$ with $x_{n+1}=a$ and $x_{n+2}=b$. For any $a<s<\min\{x_{n-1},a/2\}$ meeting a condition to be set below, let $\mathbf{y}$ be defined with $y_j=x_j$ for all $j \leq n$,

$$y_{n+1}=s, \quad y_{n+2}=b, \quad y_{n+3}=a.$$

Then we have

$$X(\mathbf{y},t)-X(\mathbf{x},t)=\begin{cases} 0 & \text{if } s<t<x_n; \\ 2(b-s) & \text{if } a<t<s; \\ -2(s-a) & \text{if } x_n<t<b. \end{cases}$$

Thus, taking $D = X_f(\mathbf{x}) - X_f(\mathbf{y})$, we have

$$\begin{aligned} D = &\int_{x_n}^{b} f(2s_{n+1}(\mathbf{x}) + t)dF(t) - \int_{x_n}^{b} f(2s_{n+1}(\mathbf{x}) + 2a - 2s + t)dF(t) \\ &+ \int_{a}^{s} f(2s_n(\mathbf{x}) - t)dF(t) - \int_{a}^{s} f(2s_n(\mathbf{x}) + 2b - 2s - t)dF(t) \\ > &\int_{x_n}^{b} 2(s-a)f'_+(2s_{n+1}(\mathbf{x}) + 2a - 2s + t)dF(t) \\ &- \int_{a}^{s} 2(b-s)f'_-(2s_n(\mathbf{x}) - 2s + 2b - t)dF(t) \\ > &\, 2(s-a)f'_+(2s_{n+1}(\mathbf{x}) + a + x_n)(F(b) - F(x_n)) \\ &- 2(b-a)f'_-(2s_n(\mathbf{x}) - 3a + 2b)F(s) \\ > &\, 0 \ \text{ if} \end{aligned}$$

$$\frac{F(s)}{s-a} < \frac{f'_+(2s_{n+1}(\mathbf{x}) + a + x_n)(F(b) - F(x_n))}{(b-a)f'_-(2s_n(\mathbf{x}) - 3a + 2b)}.$$

Note that $2s_{n+1}(\mathbf{x}) + a + x_n = 2s_n(\mathbf{x}) - a + x_n \geq -a > 0$ so the right hand side is positive. Thus choosing s strictly between a and $\min\{x_{n-1}, a/2\}$ so that the inequality holds, we have $X_f(\mathbf{x}) - X_f(\mathbf{y}) > 0$, contrary to the assumed minimality of $X_f(\mathbf{x})$. Similarly, if F is skinny at b, there can be no optimal search strategy $\mathbf{x}$ with $x_n = b$ and $x_{n+1} = a$. Therefore, if F is skinny at both a and b, there can be no terminating search strategy and the theorem now follows. ∎

The particular case of Theorem 5.1 when F is the triangular distribution was proved in [7]. The proof of Theorem 6.1 in the next section shows that when $f(t) = t^\alpha$ we have the following stronger result.

THEOREM 5.2: *Let $f(t) = t^\alpha$ where $\alpha \geq 1$ and F be compact with support $[a, b]$ where $a < 0 < b$. Then there is a non-terminating minimizing search strategy if and only if F is skinny at both a and b.*

6. Lopsided distributions

In this section we consider distributions for which precisely one of x^- and x^+ is infinite. For such F and $f(t) = t^\alpha$ where $\alpha \geq 1$ we find some necessary and some sufficient conditions for the optimal search strategies to be terminating. In particular we shall prove that in order to have a non-terminating optimal search strategy, the distribution must be skinny at the non-infinite member of $\{x^-, x^+\}$, and we shall also show that this condition is not sufficient.

Definition: We say that a distribution F is **lopsided** to the right if there is a real a such that $F(a) = 0$ and $0 < F(t) < 1$ for all $t > a$. We define F as lopsided to the left in an analogous manner.

The linear search problem for a distribution F lopsided to the right is interesting only if $a < 0$, with similar comments applying to the left. We will normalize all distributions lopsided to the right by first moving them to the right a distance $|a|$ and then rescaling them so that the starting point of the search is at 1:

$$\bar{F}(t) = F(a - at).$$

Those lopsided to the left are treated similarly. We will use $S_n(\mathbf{x})$ to denote $\sum_{i=-\infty}^{n} |1 - x_i|$ for all integers n and $G(t)$ for $1 - F(t)$ for all real t.

THEOREM 6.1: *If $f(t) = t^\alpha$ for all $t \geq 0$ where $\alpha \geq 1$, $F(0) = 0$, $F(1) < 1$ and $F(t) > 0$ for all $t > 0$, F is fat at 0, and $\int_{-\infty}^{\infty} |t - 1|^\alpha dF(t) < \infty$, then, for the search problem starting at 1, every search strategy $\mathbf{x}$ minimizing $X_\alpha(\mathbf{x})$ is terminating.*

Proof: Suppose, contrary to the assertion, that F satisfies the hypotheses and that $\mathbf{x}$ is a non-terminating search strategy which minimizes $X_\alpha(\mathbf{x})$.

Since F is fat at 0, there are $M > 0$ and $\delta > 0$ such that $F(t)/t > M$ for $0 < t < \delta$. Put $\epsilon = M/(2\alpha)$. Now

$$\infty > X_\alpha(\mathbf{x}) = \sum_{j=-\infty}^{\infty} \left| \int_{x_{j-1}}^{x_{j+1}} (2S_j(\mathbf{x}) + |t - 1|)^\alpha dF(t) \right|.$$

Thus we can find an n such that

$$\sum_{j=n}^{\infty} \left| \int_{x_{j-1}}^{x_{j+1}} (2S_j(\mathbf{x}) + |t - 1|)^\alpha dF(t) \right| < \epsilon,$$

$$x_n - x_{n+1} > 1 \quad \text{and} \quad x_{n+1} < \delta.$$

Notice that $x_n > 1$ and $S_{n+1}(\mathbf{x}) > 1$. It follows that

$$\int_{x_n}^{\infty} (2S_{n+1}(\mathbf{x}) + t - 1)^\alpha dF(t) < \epsilon$$

and

$$\int_{x_n}^{\infty} (2S_{n+1}(\mathbf{x}) + 2 + t - 1)^\alpha dF(t) < \int_{x_n}^{\infty} 2^\alpha (2S_{n+1}(\mathbf{x}) + t - 1)^\alpha dF(t) < 2^\alpha \epsilon.$$

Define the search strategy $\mathbf{y}$ by

$$y_j = x_j \text{ for all } j \le n, \quad y_{n+1} = 0, \quad y_{n+2} = \infty.$$

Then

$$X_\alpha(\mathbf{y}) = \sum_{j=-\infty}^{n-1} \left| \int_{x_{j-1}}^{x_{j+1}} (2S_j(\mathbf{x}) + |t-1|)^\alpha dF(t) \right| + \int_0^{x_{n-1}} (2S_n(\mathbf{x}) + 1 - t)^\alpha dF(t) + \int_{x_n}^\infty (2S_n(\mathbf{x}) + 2 + t - 1)^\alpha dF(t).$$

Thus

$$\begin{aligned} X_\alpha(\mathbf{x}) - X_\alpha(\mathbf{y}) > & \int_{x_n}^\infty (2S_n(\mathbf{x}) + 2 - 2x_{n+1} + t - 1)^\alpha dF(t) \\ & - \int_{x_n}^\infty (2S_n(\mathbf{x}) + 2 + t - 1)^\alpha dF(t) \\ & + \int_0^{x_{n+1}} (2S_n(\mathbf{x}) + 2x_{n+2} - 2x_{n+1} + 1 - t)^\alpha dF(t) \\ & - \int_0^{x_{n+1}} (2S_n(\mathbf{x}) + 1 - t)^\alpha dF(t). \end{aligned}$$

We see that for each $t > x_n$,

$$\begin{aligned} (2S_n(\mathbf{x}) + 2 + t - 1)^\alpha - & (2S_n(\mathbf{x}) + 2(1 - x_{n+1}) + t - 1)^\alpha \\ & = 2\alpha x_{n+1}(2S_n(\mathbf{x}) + 2 + t - 1 - 2h(t)x_{n+1})^{\alpha-1} \\ & < 2\alpha x_{n+1}(2S_n(\mathbf{x}) + 2 + t - 1)^{\alpha-1} \end{aligned}$$

for some value of $h(t)$ between 0 and 1. Thus,

$$\begin{aligned} \int_{x_n}^\infty (2S_n(\mathbf{x}) + 2 + t - 1)^\alpha - (2S_n(\mathbf{x}) + 2(1 - x_{n+1}) + t - 1)^\alpha dF(t) \\ < 2\alpha x_{n+1} \int_{x_n}^\infty (2S_n(\mathbf{x}) + 2 + t - 1)^{\alpha-1} dF(t) \\ < 2\alpha x_{n+1} \int_{x_n}^\infty (2S_n(\mathbf{x}) + 2 + t - 1)^\alpha dF(t) \\ < 2^{\alpha+1}\alpha\epsilon x_{n+1} = 2^\alpha M x_{n+1}. \end{aligned}$$

On the other hand

$$\begin{aligned} \int_0^{x_{n+1}} (2S_n(\mathbf{x}) + 2x_{n+2} - 2x_{n+1} + 1 - t)^\alpha - (2S_n(\mathbf{x}) + 1 - t)^\alpha dF(t) \\ > \int_0^{x_{n+1}} 2^\alpha dF(t) = 2^\alpha F(x_{n+1}). \end{aligned}$$

It follows that $X_\alpha(\mathbf{x}) - X_\alpha(\mathbf{y}) > 0$ if $F(x_{n+1}) > Mx_{n+1}$, which follows from $x_{n+1} < \delta$ and the definitions of M and δ. ∎

THEOREM 6.2: *If $p > 0$, F is a (normalized) distribution lopsided to the right, $f(t) = t$ for all t,*

$$\liminf_{t\to 0} F(t)/t^{1+p} < 1 \quad \text{and} \quad t^{1+1/p}G(t) \to 0 \quad \text{as } t \to \infty,$$

then every minimizing search strategy is non-terminating.

Proof: Suppose, contrary to the demonstrandum, that $\mathbf{x}$ is a minimizing search strategy that terminates. We may assume $x_i \notin \{0, \infty\}$ for all $i \leq n$, $x_{n+1} = 0$ and $x_{n+2} = \infty$. Define $\mathbf{y}$ by $y_j = x_j$ for $j = 1, \ldots, n$, $y_{n+3} = 0$ and $y_{n+4} = \infty$, with y_{n+1} and y_{n+2} still to be defined. Put $S = \sum_{j=-\infty}^{n} |x_j - 1|$. Then

$$\begin{aligned}
X(\mathbf{x}) - X(\mathbf{y}) = &\int_0^{x_{n-1}} 2S + (1-t)dF(t) + \int_{x_n}^{\infty} 2S + 2 + (t-1)dF(t) \\
&- \int_{y_{n+1}}^{x_{n-1}} 2S + (1-t)dF(t) \\
&- \int_{x_n}^{y_{n+2}} 2S + 2(1-y_{n+1}) + (t-1)dF \\
&- \int_0^{y_{n+1}} 2S + 2(y_{n+2} - y_{n+1}) + (1-t)dF(t) \\
&- \int_{y_{n+2}}^{\infty} 2S + 2(y_{n+2} - y_{n+1}) + 2 + (t-1)dF \\
= &- \int_0^{y_{n+1}} 2(y_{n+2} - y_{n+1})dF(t) + \int_{x_n}^{y_{n+2}} 2 - 2(1 - y_{n+1})dF \\
&- \int_{y_{n+2}}^{\infty} 2(y_{n+2} - y_{n+1})dF \\
= &\ 2y_{n+1}(G(x_n) - G(y_{n+2})) - 2(y_{n+2} - y_{n+1})F(y_{n+1}) \\
&- 2(y_{n+2} - y_{n+1})G(y_{n+2}) \\
= &\ y_{n+1}G(x_n) - 2(y_{n+2} - y_{n+1})F(y_{n+1}) + y_{n+1}G(x_n) \\
&- 2y_{n+2}G(y_{n+2}) \\
> &\ 0 \quad \text{if}
\end{aligned}$$

$$y_{n+1} > \frac{2y_{n+2}G(y_{n+2})}{G(x_n)} \quad \text{and} \quad \frac{F(y_{n+1})}{y_{n+1}} < \frac{G(x_n)}{2y_{n+2}},$$

this last occurring if

$$F(y_{n+1}) < y_{n+1}^{1+p} \quad \text{and} \quad G(x_n)/(2y_{n+2}) > y_{n+1}^{p}.$$

We note that if y_{n+2} is chosen large enough, then

$$G(x_n)^{1+1/p} > (2y_{n+2})^{1+1/p}G(y_{n+2}),$$

which gives us

$$\frac{2y_{n+2}G(y_{n+2})}{G(x_n)} < \left(\frac{G(x_n)}{2y_{n+2}}\right)^{1/p}.$$

Furthermore as y_{n+2} increases, these limits decrease and there are infinitely many values of y_{n+2} for which we have y_{n+1} lying between

$$2y_{n+2}G(y_{n+2})/G(x_n) \quad \text{and} \quad \Big(G(x_n)/(2y_{n+2})\Big)^{1/p}$$

for which $F(y_{n+1}) < y_{n+1}^{1+p}$. Choosing those values for y_{n+2} and y_{n+1}, we see that $X_1(\mathbf{y}) < X_1(\mathbf{x})$, so that $\mathbf{x}$ cannot be minimal. The theorem now follows. ∎

COROLLARY 6.3: *If $p > 0$, F is a (normalized) distribution lopsided to the right, $f(t) = t$ for all t, $\lim_{t\to 0} F(t)/t^{1+p} < 1$, and $\liminf_{t\to\infty} t^{1+1/p}G(t) = 0$, then every optimal search strategy is non-terminating.*

Proof: Same proof, *mutatis mutandis.* ∎

Using the Corollary with $p = 2$ it is easy to see that, for the lopsided distribution given by $F(t) = \exp(-1/t^2)$ for all $t \geq 0$, every optimal search strategy is non-terminating when $f(t) = t$ for all t.

THEOREM 6.4: *If $f(t) = t$ for all t, F is a (normalized) distribution lopsided to the right and $\mathbf{x}$ is a non-terminating optimal search strategy, then for every $x_n > 1$, we have both*

$$x_{n+1}G(x_n) > (x_{n+2} - x_{n+1})F(x_{n+1})$$

and

$$x_{n+1}G(x_n) > (x_{n+2} - x_{n+3})G(x_{n+2}).$$

Proof: Define $\mathbf{y} = \{\ldots, x_n, 0, \infty\}$ where $x_n > 1$. Then set $u = x_{n+1}$, $v = x_{n+2}$

and $w = x_{n+3}$. Taking $S_m = \sum_{j=-\infty}^{m} |x_j - 1|$ and $D = X_1(\mathbf{x}) - X_1(\mathbf{y})$, we have

$$\begin{aligned}
D = & \int_u^{x_{n-1}} 2S_n + (1-t)dF(t) \ + \int_{x_n}^{v} 2S_n + 2(1-u) + (t-1)dF(t) \\
& + \int_w^u 2S_n + 2(v-u) + (1-t)dF(t) \\
& + \int_v^{x_{n+4}} 2S_n + 2(v-u) + 2(1-w) + (t-1)dF(t) \\
& + \sum_{j=n+4}^{\infty} \left| \int_{x_{j-1}}^{x_{j+1}} 2S_j + |t-1|dF(t) \right| - \int_0^{x_{n-1}} 2S_n + (1-t)dF(t) \\
& - \int_{x_n}^{\infty} 2S_n + 2 + (t-1)dF(t) \\
> & \int_u^{x_{n-1}} 2S_n + (1-t)dF(t) \ + \int_{x_n}^{v} 2S_n + 2(1-u) + (t-1)dF(t) \\
& + \int_0^u 2S_n + 2(v-u) + (1-t)dF(t) \\
& + \int_v^{\infty} 2S_n + 2(v-u) + 2(1-w) + (t-1)dF(t) \\
& - \int_0^{x_{n-1}} 2S_n + (1-t)dF(t) - \int_{x_n}^{\infty} 2S_n + 2 + (t-1)dF(t) \\
= & \int_0^u 2S_n + 2(v-u) + (1-t)dF(t) \ - \int_0^u 2S_n + (1-t)dF(t) \\
& - \int_{x_n}^{v} 2u dF(t) - \int_v^{\infty} 2S_n + 2 + (t-1)dF(t) \\
& + \int_v^{\infty} 2S_n + 2(v-u) + 2(1-w) + (t-1)dF(t) \\
= & \, 2(v-u)F(u) - 2u(G(x_n) - G(v)) + \int_v^{\infty} 2(v-u) - 2w dF(t) \\
= & \, 2(v-u)F(u) + 2(v-w)G(v) - 2uG(x_n) \\
> & \, 0 \ \text{ if}
\end{aligned}$$

$$2uG(x_n) \leq \max\{2(v-w)G(v),\ 2(v-u)F(u)\},$$

i.e. unless both $uG(x_n) > (v-w)G(v)$ and $uG(x_n) > (v-u)F(u)$. ∎

We now provide an example of a lopsided distribution which is skinny at 0 but has no non-terminating optimal search strategy.

Example: Let $f(t) = t$ for all t and define F by

$$F(t) = \begin{cases} t^{3/2}/2 & \text{if } 0 < t \leq 1; \\ 1 - t^{-3/2}/2 & \text{if } t \geq 1. \end{cases}$$

Then $F'(t) = 3t^{-5/2}/4$ for $t \geq 1$, so that $\mu < \infty$. Let $\mathbf{x}$ be an optimal search strategy and choose n so that $x_n > 2$. Define $u = x_{n+1}$, $v = x_{n+2}$ and $w = x_{n+3}$ as before. Of the two inequalities in Theorem 6.4 the first gives

$$ux_n^{-3/2}/2 > (v-u)u^{3/2}/2$$

while the second gives

$$ux_n^{-3/2}/2 > (v-w)v^{-3/2}/2.$$

Since $v > 2$ and $1 > u > w$, $v - u$ and $v - w$ are both bigger than $v/2$, so that the inequalities assure

$$ux_n^{-3/2} > u^{3/2}v/2 \quad \text{and} \quad ux_n^{-3/2} > v^{-1/2}/2.$$

These give $2 > x_n^{3/2}u^{1/2}v$ and $2uv^{1/2}x_n^{-3/2} > 1$.

Combining,

$$4uv^{1/2} > x_n^3 u^{1/2} v, \quad \text{or} \quad 4\sqrt{x_{n+1}} > x_n^3\sqrt{x_{n+2}}$$

which is manifestly false since $x_{n+2} > x_n > 2$.

References

[1] A. Beck, *On the linear search problem*, Israel Journal of Mathematics **2** (1964), 221–228.

[2] A. Beck, *More on the linear search problem*, Israel Journal of Mathematics **3** (1965), 61–70.

[3] A. Beck and D. J. Newman, *Yet more on the linear search problem*, Israel Journal of Mathematics **8** (1970), 419–429.

[4] A. Beck and P. Warren, *The return of the linear search problem*, Israel Journal of Mathematics **14** (1973), 503–512.

[5] A. Beck and M. Beck, *Son of the linear search problem*, Israel Journal of Mathematics **48** (1984), 109–122.

[6] A. Beck and M. Beck, *The linear search problem rides again*, Israel Journal of Mathematics **53** (1986), 365–372.

[7] A. Beck and M. Beck, *The revenge of the linear search problem*, SIAM Journal on Control and Optimization **30** (1992), 112–122.

[8] D. S. Mitrinović, *Analytic Inequalities*, Springer-Verlag, Berlin, 1970.

[9] A. W. Roberts and D. E. Varberg, *Convex Functions*, Academic Press, New York, 1973.